EFFECT OF VISCOELASTIC PROPERTIES OF A LIQUID ON THE DYNAMICS OF SMALL OSCILLATIONS OF A GAS BUBBLE

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UDC 532.5:532.13

A series of papers has been devoted to questions of gas bubble dynamics in viscoelastic liquids. Of these papers we mention [1-4]. The radial oscillations of a gas bubble in an incompressible viscoelastic liquid have been studied numerically in [1, 2] using Oldroyd's model [5]. An exact solution was found in [3], and independently in [4], for the equation of small density oscillations of a cavity in an Oldroyd medium when there is a periodic pressure change at infinity. The analysis of bubble oscillations in a viscoelastic liquid is complicated by properties of limiting transitions in the rheological equation of the medium. These properties are of particular interest for the problem under investigation. These properties are discussed below, and characteristics of the small oscillations of a bubble in an Oldroyd medium are investigated on the basis of a numerical analysis of the exact solution obtained in [3].

The fundamental characteristics of the small oscillations of a bubble in a viscoelastic liquid, obtained in [3], on being reduced to dimensionless form are

$$D = \left\{\frac{\left[\omega(\omega^2 - b) + (\lambda_1^{-1} - s_1)\, 2\omega\delta\right]^2 + \left[(a\omega^2 - c) + (\lambda_1^{-1} - s_1)(\omega^2 - \delta^2 - \mu^2)\right]^2}{\left[4\omega^2\delta^2 + (\omega^2 - \delta^2 - \mu^2)^2\right]\left[\omega^2(\omega^2 - b)^2 + (a\omega^2 - c)^2\right]}\right\}^{1/2}; \tag{1}$$

$$\operatorname{tg}\alpha = -\frac{2\omega\delta\left[\omega^2(\omega^2 - b)^2 + (a\omega^2 - c)^2\right] + \omega(\omega^2 - b)(\lambda_1^{-1} + s_1)\left[4\omega^2\delta^2 - (\omega^2 - \delta^2 - \mu^2)^2\right]}{(\omega^2 - \delta^2 - \mu^2)\left[\omega^2(\omega^2 - b)^2 + (a\omega^2 - c)^2\right] + (\lambda_1^{-1} - s_1)(a\omega^2 - c)\left[4\omega^2\delta^2 + (\omega^2 - \delta^2 - \mu^2)^2\right]},$$

$$a = \lambda_1^{-1} + 4\eta_0\lambda_2\lambda_1^{-1}, \quad b = 3k + 2\sigma(3k - 1) + 4\eta_0\lambda_1^{-1},$$

$$c = \lambda_1^{-1}\left[3k + 2\sigma(3k - 1)\right], \quad s_1 = A + B - a/3,$$

$$\mu = \sqrt{3}(A - B)/2, \quad \delta = -1/2(A + B) - a/3, \quad A = (-q/2 + \sqrt{Q})^{1/3},$$

$$B = (-q/2 - \sqrt{Q})^{1/3}, \quad Q = (p/3)^3 + (q/2)^2, \quad q = 2(a/3)^3 - 1/3ab + c,$$

$$p = b - 1/3a^2, \quad \sigma = \sigma^*(R_0^* p_\infty^*)^{-1}, \quad p_0 = p_0^*/p_\infty^*,$$

$$\eta_0 = \eta_0^*(\rho^* p_\infty^*)^{-1/2}/R_0^*, \quad T^* = R_0^*(\rho^*/p_\infty^*)^{1/2},$$

$$\omega = \omega^* T^*, \quad \lambda_1 = \lambda_1^*/T^*, \quad \lambda_2 = \lambda_2^*/T^*,$$

where D is the amplitude of small density oscillations, relative to the acoustic pressure p_0; α is the phase shift between the acoustic pressure and the forced density oscillations; $-s_1$ and $-\delta$ are the damping coefficients of characteristic oscillations of density with a frequency μ; ρ^*, η_0^*, λ_1^*, λ_2^* are the density, viscosity, relaxation time, retardation time, and surface tension of the liquid, respectively; ω^* is the acoustic pressure frequency; p_∞^* is the pressure at infinity; k is the polytropy exponent; and R_0^* is the initial radius of the bubble. Dimensional quantities are denoted by an asterisk.

Equations (1) correspond to the case when the characteristic equation for small density oscillations in a viscoelastic liquid has a positive discriminant Q [3]:

$$s^3 + as^2 + bs + c = 0. \tag{2}$$

Voronezh. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 3, pp. 82-87, May-June, 1976. Original article submitted June 18, 1975.

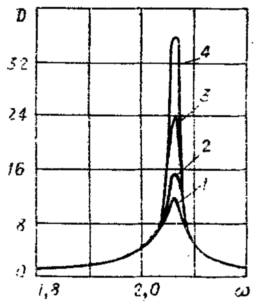

Fig. 1

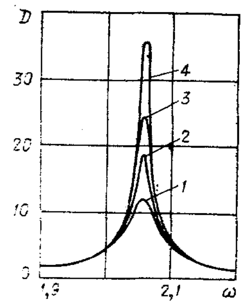

Fig. 2

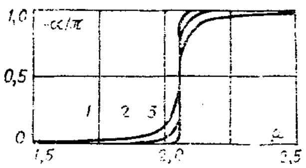

Fig. 3

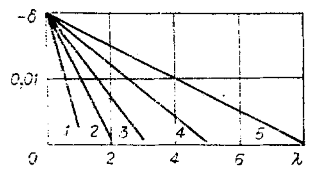

Fig. 4

This case is of most interest, since it corresponds to the underdamped regime of characteristic bubble oscillations.

In order to investigate the effect of the parameters λ_1 and λ_2 on the nature of small density oscillations, numerical calculations of Eqs. (1) were made on a model BÉSM-4 computer for $\eta_0 = 0.01$; $\sigma = 0.001$; $k = 1.4$. The results are given in Figs. 1-7.

Resonance curves for a bubble in a Maxwellian liquid ($\lambda_2 = 0$) are given in Fig. 1 (1-4 correspond to $\lambda_1 =$ 0, 0.25, 0.5, 0.75). It can be seen that as λ_1 increases the resonance amplitude of the oscillations increases sharply. For large values of relaxation time the graph of the function $D = D(\omega)$ approaches a discontinuous curve, characteristic of an ideal liquid [6]. This is explained by the fact that when $\lambda_1 \to \infty$ Oldroyd's equation

$$\tau_{ik} + \lambda_1 D\tau_{ik}/Dt = 2\eta_0(e_{ik} + \lambda_2 De_{ik}/Dt) \tag{3}$$

assumes the form $\tau_{ik} = \text{const}$. Here τ_{ik} and e_{ik} denote the excess stress tensor and the deformation rate tensor, respectively. Since the cavity was initially at rest, we can take $\tau_{ik} = 0$. This also means that the stress tensor for the liquid becomes spherical.

It also follow from Fig. 1 that the resonance curves 2-4 for a Maxwellian liquid differ from curve 1 corresponding to a Newtonian liquid only in a narrow zone close to the resonance frequency ω_r. For $\omega \gg \omega_r$ and $\omega \ll \omega_r$ all the curves merge into one.

The functions $D = D(\omega)$ are shown in Fig. 2 for an Oldroyd medium with $\lambda_1 = 0.75$, curves 1-4 correspond to the values $\lambda_2 = 0.75$, 0.4, 0.2, 0. It is clear that a retardation of deformation rates (as opposed to stress relaxation) decreases the amplitude of bubble oscillations. For $\lambda_2 = \lambda_1$ the resonance curve assumes the same form as for a Newtonian liquid. In this case Oldroyd's equation describes a normal viscous liquid. In fact, for a Newtonian liquid we have

$$\tau_{ik} = 2\eta_0 e_{ik}. \tag{4}$$

Differentiating Eq. (4) with respect to time, multiplying by an arbitrary parameter λ, and combining it with the initial equation, we obtain Eq. (3) with $\lambda_1 = \lambda_2 = \lambda$. This property of an Oldroyd medium was not noted in [2], in particular, where a series of calculations was carried out for different values of λ_1 and λ_2 which were, however, equal to each other. In this case identical curves; corresponding to a Newtonian liquid, were obtained.

Thus, for all values of λ_2, satisfying the inequality $\lambda_1 > \lambda_2 > 0$, the resonance curve for an Oldroyd liquid is contained between two limiting curves corresponding to a Maxwellian and a Newtonian liquid.

The requirement of a positive-definite entropy output in an Oldroyd medium (see [7]) leads to the condition $\lambda_1 > \lambda_2 > 0$. The calculations given in [4] in graphical form for the cases $\lambda_2 > \lambda_1 > 0$ do not satisfy this condition. At the same time Oldroyd's equation permits a limiting transition [3] for $\lambda_1 \to 0$ to the rheological equation of a viscoelastic liquid with a retardation of deformation rates, for which the parameter λ_2 must be taken positive. This fact has recently been proved strictly in [8] on the basis on a thermodynamic analysis. The equation with negative λ_2, known in the literature as Walters' equation of state [9], does not describe any real liquid, for the reasons given in [8].

The phase shift angle α is given in Fig. 3 as a function of the acoustic pressure ω for a bubble in a Maxwellian medium. Curves 1-3 correspond to $\lambda_1 = 0$, 1, 10. It is clear that as λ_1 becomes larger the nature of the phase shift approaches a step-function at the frequency $\omega = \omega_r$ corresponding to an ideal liquid.

In connection with the fact that an Oldroyd medium passes to a Newtonian liquid for $\lambda_1 = \lambda_2$, it is convenient to treat the quantity λ_1 and the difference $\lambda_1 - \lambda_2 = \lambda$ as parameters which characterize the medium. The damping coefficient δ is shown in Fig. 4 as a function of the parameter λ; the curves 1-5 correspond to $\lambda_1 = 1$, 2, 3, 5, 8. All the curves emerge from the one point, since the value $\lambda = 0$ corresponds to a Newtonian liquid. It is clear that the damping coefficient is a practically linear function of the parameter λ, while the quantity $-\delta$ decreases as λ increases. The curves terminate at the point $\lambda = \lambda_1$ on the abscissa, since taking the calculations further would correspond to negative values of λ_2. For a fixed value of λ an increase of λ_1 leads to an increase in the damping coefficient. However, calculations show that for small values of λ_1 and λ an Oldroyd medium is completely characterized by the single parameter λ.

The damping coefficient $-\delta$ is shown in Fig. 5 as a function of λ_1 for fixed values of λ_2. Curves 1-5 correspond to $\lambda_2 = 0$, 0.5, 1, 2, 5. For $\lambda_1 \to \infty$ the quantity $\delta \to 0$, which corresponds to an ideal liquid. Increasing the retardation time λ_2 for a constant value of λ_1 leads to an increase in the damping coefficient. However, the damping coefficient in a viscoelastic liquid is always less than in a Newtonian liquid. We also note that as λ_2 is increased the curve giving the damping coefficient as a function of the relaxation time takes on a more gently sloping character.

The second damping coefficient $-s_1$, specific for a relaxing medium, is given in Fig. 6a as a function of λ_1. Calculations show that the quantity s_1 depends only negligibly on the retardation time λ_2. As λ_1 increases without limit the damping coefficient $-s_1$ decreases monotonically to zero.

It is clear from Figs. 4-6 that the damping coefficients $-\delta$ and $-s_1$ are positive for all values of λ_1 and λ_2. In this connection we should mention an inaccuracy appearing in [4] where the character of small oscillations of a cavity in a viscoelastic medium is analyzed. It is shown in [4] that under certain conditions the solution of the equation for small oscillations of a cavity in an Oldroyd liquid can increase without limit. In the opinion of Yang and Lawson this signifies the onset of cavitation in the small amplitude acoustic field. For this to happen the quantities δ or s_1, representing the real parts of the roots of the characteristic equation (2), must be positive. Inequalities are formulated in [4] which the parameters of the problem must satisfy in this case. It can be shown that Eq. (2) has no roots with positive real parts. Actually, the well known Routh–Hurwitz conditions [10], which are necessary and sufficient for all the roots of Eq. (2) to have negative real parts, can be written in the form

$$b > 0, \quad c > 0, \quad ab > c.$$

The first two inequalities are satisfied in an obvious manner, while the last is equivalent to the relation

$$4\eta_0\lambda_1^{-1}\{\lambda_1^{-1} + \lambda_2[3k + 2\sigma(3k - 1) + 4\eta_0\lambda_1^{-1}]\} > 0,$$

which is also always satisfied. Thus, there are no physically realizable conditions under which the criteria for the onset of cavitation, formulated in [4], can be satisfied. The damping coefficient $-\delta$ and $-s_1$ are always positive, which was pointed out in [3]. We note that the possibility of the unlimited growth of a bubble in a small amplitude acoustic field was derived in [11], and there is a reference to this in [4]. In [11] this derivation depends on the collision of certain parameters of the problem, and on an error in linearizing the equation for the bubble radius.

The phase shift angle α is shown in Fig. 6b as a function of λ_1 for a fixed value of the frequency $\omega = 2$, and the curves 1-4 correspond to $\lambda_2 = 0$, 1, 2, 5. It is clear that as the relaxation time λ_1 increases, the phase shift between the cavity oscillations and the pressure oscillations at infinity decreases monotonically for a given frequency ω. The curve $\alpha = \alpha(\lambda_1)$ is steepest in a Maxwellian liquid (curve 1), while for $\lambda_2 > 0$ the curve becomes more gently sloping. An increase in retardation time leads to an increase in the phase shift. However, for all values of λ_2 the phase shift in an Oldroyd liquid is less than in a Newtonian liquid. We note that this applies only to frequencies up to the resonance frequency ($\omega < \omega_r$). For $\omega > \omega_r$ the opposite result holds (see Fig. 3).

The characteristic frequency of the oscillations μ and their amplitude D are given in Fig. 7a, b as functions of the relaxation time λ_1 up to the resonance frequency $\omega = 2$. Curves 1-5 correspond to $\lambda_2 = 0$, 0.5, 1, 2, 5.

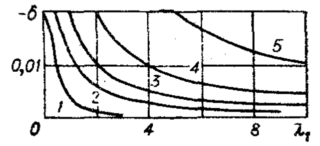

Fig. 5

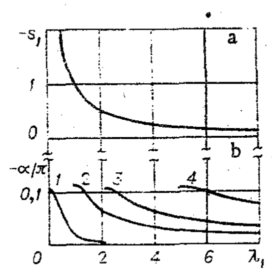

Fig. 6

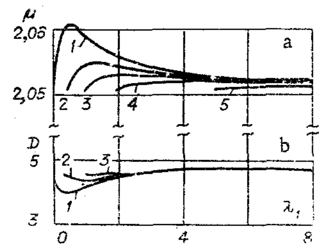

Fig. 7

It is clear that as λ_1 increases the characteristic frequency of the oscillations increases at first, and then decreases monotonically. Retardation has the effect of reducing the magnitude of μ. We note that the characteristic frequency of bubble oscillations in an Oldroyd liquid is always larger than in a Newtonian liquid, but less than in a Maxwellian liquid.

The data of Fig. 7b allow us to illustrate some properties of the variation of oscillation amplitude up to the resonance frequency as the parameters λ_1 and λ_2 are varied. These properties could not be represented in the scale chosen in Figs. 1 and 2. It is clear that as λ_1 increases, the amplitude D first decreases, and only then increases monotonically, approaching the amplitude of bubble oscillations in an ideal fluid at the frequency $\omega=2$ when λ_1 becomes large. This can apparently be explained by the fact that for small values of λ_1 the increase in the characteristic oscillation frequency μ (Fig. 7a) and, consequently, the displacement of the resonance curve into the region of large frequencies, occurs more rapidly than the increase in oscillation amplitude. Increasing the retardation time in this region of variation of λ_1 leads to an increase in the amplitude.

Thus, the linear analysis given above shows that the oscillations of bubbles in viscoelastic fluids close to the resonance frequencies should develop more strongly than in Newtonian liquids.

The authors are grateful to V. N. Nikolaevskii for useful advice and for discussing the results.

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